Fixed Income
Problem set and notes to prepare for second quiz on the Vasicek model.

Problems: Show all work. Clearly indicate your answer to each question. Practice answering precisely and concisely.

Last week we introduced the model for the dynamics of the instantaneous short rate in the Vasicek model. And we saw that the price of a 0-coupon bond that pays \$1 on date T is equal to: $E\left[e^{-\int_t^T r_\tau d\tau}\right]$. That is for a given interest rate path – between t and T – we obtain the sum of the instantaneous rate, and the bond's price is equal to e raised to this power. The $E[\ ]$ part takes the average of this price over all possible paths–governed by the laws of probability. For Quiz 11, we used the computer to generate a path of the instantaneous rate–or state variable–(the inner loop), and then Monte Carlo to compute the expectation (the outer loop).

This is a one-factor model because the only thing that can change from one day to the next is the level of the instantaneous short rate (i.e., the factor).

The next step for Quiz 12 is to add the concept of risk aversion to the model. In the term structure setting the concept of risk aversion is linked to a planning horizon. If I want to guarantee a nominal \$ amount to be on hand in 10 years, then lending for 6 months–with the intent to re-invest on exipry–is riskier than lending for 10 years. This could give rise to a solidity premium. By contrast, if we want the highest nominal return over any period, then lending for a longer term is riskier since rates could rise *unexpectedly* in the future. This could give rise to a liquidity premium. Note that the *unexpected* adverb is critical here. This is because in the Vasicek model, if the current state variable is below its long-run mean then the yield curve will be upward sloping, and the 10 year STRIPS will have a higher yield to maturity than the 6-month STRIPS.

So the question of whether there is a risk premium in the yield curve, and what form it takes is an empirical one. –We have to look at the data. In this context, we historically have seen a liquidity premium. In the 2010's (post Global Financial Crisis), the liquidity premium seems fairly small by the standard of the second half of the 20th Century.

A risk premium in the Vasicek model introduces a distinction between the physical process that generates the path of the state variable and the process that we place inside the integral: $E\left[e^{-\int_t^T r^*_\tau d\tau}\right]$. r^* is the short rate in the equivalent risk-neutral world– the "world" used to get the prices of risky assets, where:

$$dr^* = k(\theta - r^*)dt + \lambda \cdot dt + \sigma \cdot d\omega$$

Alternately (to see it in the context of the long-run mean):

$$dr^* = k(\theta + \frac{\lambda}{k} - r^*)dt + \sigma \cdot d\omega$$

So we can rewrite the process in the equivalent risk-neutral world:

$$dr^* = k(\hat{\theta} - r^*)dt + \sigma \cdot d\omega$$

by defining $\hat{\theta} = \theta + \frac{\lambda}{k}$

Why do we we call r^* the state variable in the *equivalent risk-neutral world*? It is because when we use this as the state variable to compute the price of any asset, we are taking expectations–just like we did in the absence of risk. The result however gives us the price that accounts (properly) for risk in our world (the actual or *physical*) world. So the prices of all assets are the same in our world and this risk-neutral world – making it an equivalent world.

Note that when λ is positive long rates are higher than they would be in the absence of a risk premium – there is a liquidity premium. By contrast, if there is a solidity premium, λ is negative.

The next innovation is that because the future state variable comes from a normal distribution, we don't have to use the computer to approximate the integral from t to T– that is available analytically (in "closed form"). The same is true for the expectation of $E\left[e^{-\int}\right]$.

We don't have to memorize the formula for a 0-coupon bond in the model. We know that the model has 4 parameters and a state variable. We should understand that the shape of the yield curve depends on three things:

- Expectations
- Risk premium
- Convexity.

You should understand how each aspect of the yield curve's shape is related to each of the parameters, and to the state variable.

Quiz 12 Prep Questions

1. Write a function in VBA that takes the following arguments as inputs:
    - The level of the state variable (the instantaneous rate)
    - The model parameters:
        - θ (the long run mean of the state variable)
        - k (the pull toward the long run mean of the state variable)
        - σ (the effect of randomness on the state variable)
        - λ (the risk premium)
    - The term of a 0-coupon bond,

    and produces the yield to maturity of a 0-coupon bond with that term (as the value of the function, on output).
2. Use this function to construct the yield curve, plot the curve, and evaluate the effects of changing each of the parameters and the level of the state variable on the shape of the curve. Analysis of such changes should be couched in the context of:
    - Expectations
    - Risk premium
    - Convexity.
3. Write a function in VBA that takes the following arguments:
    - The level of the state variable (the instantaneous rate)
    - The model parameters:
        - θ (the long run mean of the state variable)
        - k (the pull toward the long run mean of the state variable)
        - σ (the effect of randomness on the state variable)

    - λ (the risk premium)
- The term of a coupon bond
- The coupon rate,

to produce the price of the coupon bond–assuming that each coupon period is exactly $\frac{1}{2}$ of a year, and the next payment is exactly $\frac{1}{2}$ a year from now.

4. To use Vasicek model in our annuity hedging context, we need the concept of factor duration. Duration tells us about the effect of a change in the yield to maturity on a security's price. We use the chain rule to get the factor duration – which is the effect of a change in the factor on a security's price. So we need to know the effect of a change in the factor on the 0-coupon bond yield. We see from inspection that B is multiplied by $\frac{r}{-T}$ which gives $\frac{dy}{dr} = \frac{1-e^{-kT}}{kT}$. Consider a 30-year 0-coupon bond when $k = 1$, for example. In this case, $\frac{dy}{dr} = .03333$, so the factor duration would be: $30 \cdot .03333 = 1$. *You should know the formula for the factor duration in the Vasicek model. And be able to provide an intuitive description of it.* The notions of portfolio factor duration follow the same logic as duration itself. Thus the factor duration of a coupon bond, for example, is the value-weighted factor duration of each of the bond's cash flows in the same way that duration is the value-weighted duration of each of the bond's cash flows

   This highlights a problem with this (and any similar) 1-factor model: there is very little volatility in long-term rates. And there is virtually no difference between factor durations for most instruments.

   Write a VBA subroutine to compute the factor duration of a 0-coupon bond. *On the quiz I can give you a code that has the bond price from the VBA model, and ask you to add a section that computes the factor duration of a 0-coupon bond.*