Dynamic duration hedging of an annuity

We consider the problem of a financial institution that provides annuities to retirees. We have a customer with a retirement portfolio that she would like to convert to a fixed-rate annuity. The contractual terms will specify the payment, the term and the number of payments per year.

Conceptually we could imagine buying a 0-coupon bond corresponding to each future cash flow – all would have the same par value. This of course determines the yield to maturity on the annuity. In practice, we may want to dynamically hedge the annuity exposure using just 2 or 3 securities.

We therefore need a computer program that will allow us to evaluate how such a hedge will perform over a variety of interest rate scenarios. We need modules of a program that compute the value of the securities that we use to hedge, make the payments on the annuity each period, and ascertain how much value the position makes or loses over the term.

We will nest these inside a do loop that goes through each period. The length of the periods is determined by the frequency of annuity payments. The period that the program covers is determined by the annuity's term.

Once we have modules that ascertain the duration of the annuity, the durations of the hedging assets, make the payment on the annuity and rebalance the hedge portfolio each period, we need to have another part of the program that produces interest rates from one period to the next.

The first procedure we will use to get these interest rates is a simple bounded arithmetic Brownian motion. It works like this: We start the time loop with the interest rates equal to the current rates. At this stage we are assuming that the yield curve is always a flat line–which is compatible with the use of duration as a risk measure.

So we start with the level of the yield curve today (date 0): y_0.

$$y_t = y_{t-1} + Z_t \cdot \sigma\sqrt{\Delta t} \tag{1}$$

$$y_t \leq K \tag{2}$$

$$y_t \geq F \tag{3}$$

$$Z_t \sim \mathcal{N}(0,1) \tag{4}$$

Here σ is the standard deviation of changes in interest rates (expressed on an annual basis) and Δt is the number of years between times t and $t+1$. K is the cap on the interest rate process and F is the floor.

The VBA subroutine IRPath simulates a path of interest rates from this process in the Simple1 worksheet of the DOOne spreadsheet, and prints out the rate at each point. It uses the inverse normal distribution to get the draw from the normal distribution. This works as follows:

1. Obtain a random draw from a Uniform(0,1) distribution.

2. This is now interpreted as a probability–which we will project onto the standard, unit normal cumulative distribution function. This function shows the probability that a draw from a normal distribution with 0 mean and standard deviation of 1 will be less than Z.

3. The probability returns a specific value of Z.

We have a loop that produces a new level of the yield curve at each point in the annuity's life. And given that level of the yield curve we get the value of the assets and liability and make the required cash flow, and then rebalance the hedge. We keep all gains and losses in each period inside the trade–by using all available assets to form the hedge.

Monte Carlo analysis repeats the entire time period for an alternative interest rate path. We take many paths, and then we can construct the distribution of profits and losses.

Convexity

The exercise above will always generate a positive value at the end of the annuity's life because the hedge portfolio is long convexity, and the yield curve exhibits only parallel shifts. This fact is important as it tells us that such an economy is not consistent with the absence of arbitrage. The reason is that no one would hold any intermediate term bond. In a world where the yield curve is always flat and experiences random parallel shifts, convexity is beneficial and costless.

This means that a proper model that explains the shape and dynamics of the yield curve must generate a cost to convexity, and/or dynamics in which a positive-convexity position loses money. The price of convexity is the butterfly spread – we would generally expect that the yield on a portfolio the was 50% invested in a 6-month bill and 50% invested in a 20-year STRIPS would have a lower yield to maturity than a bullet portfolio that is 100% invested in a 10.25-year STRIPS. Of course the value of convexity–and hence its implicit price in a proper model–is a function of how volatile interest rates will be.

We will use the Vasicek model to explain why the yield curve has the shape that it does at any point in time, and to explain the dynamics of the curve – how it changes over time. Our overarching intuition is that the shape of the yield curve – at any point in time – depends on three things:

1. Expectations
2. Convexity
3. Risk premia

1. Expectations. If we expect the short-term rate to rise then we will require longer-term rates to be high enough to compensate for the rise in the short rate over their terms. If we expect the short-term rate to fall then we will require longer-term rates to be low enough to compensate for the drop in the short rate over their terms. Finally, of we expect the short rate to be unchanged over the future, then long rates will reflect this expectation.
We model expectations using a mean-reverting random process.

2. Convexity. The more volatile interest rates are–the higher will be the cost of convexity. This drives longer term rates below their levels if there were less volatility. Volatility refers to the amount of variation in the short rate (or short rate factors).

3. Risk. If investors were risk-neutral, then so long as long rates reflect the expectation of future short rates and properly compensate for convexity, then we would be indifferent between lending (or borrowing) for 5 years using a 5-year zero-coupon bond or lending (or borrowing) for 6 months–with the expectation of rolling over the position in 6 months, 1 year, 18 months, ... , 54 months. If investors and borrowers are risk averse, then we may not be indifferent between these 2 alternatives. If we have a 5-year planning horizon (for example we are financing a 5-year project), then the former strategy is less risky (since we know what our nominal cost will be over the term) than the latter strategy.

This discussion should make it clear that risk aversion is related to the concept of a planning horizon (the duration that would minimize the lender/borrower's risk). If the economy-wide planning horizon is short-term, then risk aversion would manifest in a liquidity premium in which longer term rates have to compensate investors with a risk premium. If, on the other hand, the economy-wide planning horizon is long-term, then risk aversion would manifest in a solidity premium–in which shorter term rates have to compensate investors with a risk premium. So the question of whether there is a risk premium, and what effect it has on rates is an empirical one. For the most part, the data suggest that there is a (positive) liquidity premium that varies over time.

In this course we will not estimate the parameters of the model. Instead we will examine the effects that the parameters have on the shape and dynamics of the yield curve.