

## Fixed Income

Problem set to prepare for first quiz on the Vasicek model.

**Problems: Show all work. Clearly indicate your answer to each question.** Practice answering precisely and concisely.

1. Our first interest rate model was that the interest rate followed an arithmetic Brownian motion, with bounds:

$$\begin{aligned}r_t &= r_{t-1} + \sigma \cdot \sqrt{\Delta t} \cdot Z_t \\r_t &\geq F \\r_t &\leq C\end{aligned}$$

And that every point on the yield curve was the same. That is, the yield curve is always a horizontal line.

- (a) What is  $E(r_t|r_{t-1})$  in this model?
  - (b) What is  $Z_t$  in this expression?
  - (c) How do we get  $Z_t$  in a computer program like VBA?
  - (d) How do we impose the inequality constraints in a computer program like VBA?
  - (e) Is this model for the yield curve consistent with the absence of arbitrage? *Explain.*
2. Write down the continuous-time model for the instantaneous (short) rate of interest in the 1-factor Vasicek model.
  3. Explain how this model differs from –and is similar to– the arithmetic Brownian motion model for the short rate.
  4. Write down the equation for a zero-coupon bond price in the 1-factor Vasicek model, assuming risk-neutrality.
  5. In the context of your answer to the previous question, explain how the 1-factor Vasicek model fixes the problem that made the simple arithmetic Brownian motion model inconsistent with the absence of arbitrage.
  6. In the context of your answers to the previous two questions, explain how increasing the standard deviation of the instantaneous (short) rate of interest in the 1-factor Vasicek model affects the prices of zero-coupon bonds of various terms, and accordingly, the shape of the yield curve.
  7. Explain the way that expectations affect the shape of the yield curve in the 1-factor Vasicek model. Explain the conditions under which the yield curve will be downward sloping, upward sloping, and horizontal in the Vasicek model.

8. Below is the VBA code for simulating the instantaneous (short) rate in the 1-factor Vasicek model.

```

Sub OnePdemo()
Worksheets("Basics").Activate
  NYears = ActiveSheet.Cells(43, 2).Value
  deltat = ActiveSheet.Cells(44, 2).Value
  sigma = ActiveSheet.Cells(42, 2).Value
  R0 = ActiveSheet.Cells(39, 2).Value
  k = ActiveSheet.Cells(41, 2).Value
  theta = ActiveSheet.Cells(40, 2).Value
Application.ScreenUpdating = True
  sd = sigma * Sqr(deltat)
  ndraws = NYears / deltat
  ReDim r(0 To ndraws) As Double
  r(0) = R0
  Var = sigma * sigma
  tvar = (Var / (2 * k)) * (1 - Exp(-2 * k * deltat)) ' This is the variance of the r(t),
                                                    ' conditional on r(t-1).

  tsig = Sqr(tvar)
  For Index = 1 To ndraws
start:
    Static rand1, rand2, s1, s2, z
    rand1 = 2 * Rnd - 1
    rand2 = 2 * Rnd - 1
    s1 = rand1 * rand1 + rand2 * rand2
    If s1 > 1 Then GoTo start
    s2 = Sqr(-2 * Log(s1) / s1)
    z = rand1 * s2
    ER = theta + (r(Index - 1) - theta) * (Exp(-k * deltat)) ' This is the expected interest
                                                            ' rate at time t + delta-t conditional on r(t).

    r(Index) = ER + tsig * z
  Next Index
  For Index = 0 To ndraws
    ActiveSheet.Cells(Index + 46, 1).Value = Index
    ActiveSheet.Cells(Index + 46, 2).Value = r(Index)
  Next Index
End Sub

```

- (a) Use this code to plot a path from the process –writing the rates to a spreadsheet and making a time-series plot for the following parameters:  $\theta = .04$ ;  $k = .9$ ; and  $\sigma = .001$ . Vary the state variable  $r_0$  from 0.005 to 0.165 in increments of .01 and make note of what happens to the paths as  $r_0$  changes. Vary the other parameters – one at a time, and make note of what happens to the paths as each changes.
- (b) Evaluate the Box-Müller algorithm: take 10,000 draws of  $z$  from this algorithm, write them to Excel, and plot a histogram of them. Compute the mean and standard deviation of the draws.
- (c) Show how you have to modify the code to use the inverse CDF method instead of the Box-Müller method to get  $Z$ .

- (d) Show how you can augment the code to obtain the price of a  $T$ -period zero-coupon bond in the Vasicek model.
- (e) The code that answers the preceding question will have 2 “do loops.” Carefully explain what each loop does, and why one is nested in the other (outer) loop.
- (f) Use the model to generate the prices of 6-month, 15.5-year, and 30-year zero coupon bonds with the following parameters:

Parameter	Value
$\theta$	0.03
$\sigma$	.02
$k$	.7

And state variable,  $r_0$  equal to .025.

- i. Determine the price of the 0.5-15.5-30 butterfly spread – i.e., the difference between the yield on the 15.5-year zero-coupon bond and the yield on a portfolio containing the 6-month and 30-year zero-coupon bonds with the same duration as the 15.5-year zero-coupon bond.
  - ii. Next increase  $\sigma$  by increments of .02 to .12, and record the value of the 0.5-15.5-30 butterfly spread for each case. Explain the results.
  - iii. Next change the state variable,  $r_0$ , from .025 to 0.06, and repeat the above 2 exercises. Discuss the effects of this change in the state variable on your answers.
9. Consider two scenarios in a highly stylized setting. In scenario 1, the instantaneous interest rate over the next 10 years will not deviate much from 10%. In scenario 2, the average of the instantaneous interest rate over the next 10 years will be either be 19% or 1% – each with a 50% probability.
- (a) What is the expected value of the instantaneous interest rate path in the 2 scenarios?
  - (b) What is the 10-year spot rate in the 2 scenarios?
  - (c) What does this have to do with convexity?