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Refresher on The Calculus vis-à-vis Duration

Modified Duration

Start with the relationship between a zero-coupon bond and its bond-equivalent yield-to-maturity:

$$P = \frac{100}{(1 + \frac{y}{2})^{2T}} \tag{1}$$

Where: P is the bond's price.
100 is the bond's par value.
y is the bond's bond-equivalent yield-to-maturity (i.e., semi-annually compounded).
T is the term of the bond in years.
Modified duration is defined to be $-\frac{dP}{dy}/P$ – that is the negative of the first derivative of the price taken with respect to its yield, divided by price.
We can compute the first derivative by repeated application of the chain rule:

$$\frac{dP}{dy} = 100 \cdot \frac{1}{2} \cdot -2T \cdot \left(1 + \frac{y}{2}\right)^{-2T-1} \tag{2}$$

Notice that this can be simplified:

$$\frac{dP}{dy} = P \cdot -T \cdot \left(1 + \frac{y}{2}\right)^{-1} \tag{3}$$

So this gives us modified duration:

$$MD = \frac{T}{1 + \frac{y}{2}} \tag{4}$$

Duration

Duration looks at a similar idea, but instead of dP/dy we consider dP/dr where r is the "continuously compounded interest rate." In this case, we have:

$$P = 100e^{-rT} \tag{5}$$

Now to get dP/dr we also use the chain rule, and we remember that $\frac{de^x}{dx} = e^x$.
We have:

$$\frac{dP}{dy} = 100 \cdot -T \cdot e^{-rT} = -T \cdot P \tag{6}$$

And:

$$D = T \tag{7}$$

Now it may seem odd, that $\frac{de^x}{dx} = e^x$, so let's explore this result in the compound interest setting. We have already seem how to apply the chain rule to get $\frac{dP}{dy}$ for semi-annual compounding. Let's look at daily compounding. In this case, we have:

$$\frac{dP}{dy} = 100 \cdot \frac{1}{365} \cdot -365T \cdot \left(1 + \frac{y}{365}\right)^{-365T-1} \tag{8}$$

which simplifies to:

$$\frac{dP}{dy} = -T \cdot \frac{P}{1 + \frac{y}{365}} \tag{9}$$

So you can hopefully see that as the compounding interval shrinks, the "yield adjustment" also shrinks. And for continuous compounding, this adjustment is infinity, so we get the result, (since $1 + \frac{y}{\infty} = 1$).